Indian Statistical Institute
First Semester Examination 2004-2005
B.Math II Year
Differential Equations II

Time: 3 hrs Date:24-11-04 Max. Marks : 50

Answer all questions:

1. Let $p_0, p_1, \ldots$ be the sequence of polynomials given by

$$p_n(t) = \frac{d^n}{dt^n}(t^2-1)^n$$

(a) Show that $\int_{-1}^{1} dt \ \ t^k p_n(t) = 0$ for $k < n$ [3]

(b) Calculate $\int_{-1}^{1} dt \ [p_n(t)]^2$ [3]

2. For $t > 0$, $m^n R^n$ $f : R^n \to \mathbb{C}$ any bounded continuous function define

$$u(0,x) = (4\pi t)^{-n/2} \int_{R^n} dy\, f(y) \exp\{-[(x-y)^2/(4t)]\} = \int dy f(y) Q(t,x,y)$$

(a) Prove formally

$$\frac{\partial \mu}{\partial t} = \sum_{j=1}^{n} \frac{\partial^2 \mu}{\partial x^2}$$

[5]

(b) Prove rigourously

$$\frac{\partial u}{\partial t}(t,x) = \int dy f(y) \frac{\partial}{\partial t} Q(t,x,y)$$

[4]

(c) Take $n = 1$. Prove rigorously

$$\frac{\partial u}{\partial x}(t,x) = \int dy \ f(y) \ \frac{\partial}{\partial x} \ Q(t,x,y)$$

[7]

HINT: For b and c you can use the fact Let $g : (a,b) \times R^n \to \mathbb{C}, g = g(t,x)$ t in $(a,b), x$ in R^n be any continuous function such that

(a) $\int\limits_{R^n} dx|g(t,x)| \quad < \infty$

$\frac{\partial g}{\partial t}(t,x)$ exists, continuous and for each t_0 in (a,b) there exists $\partial > 0$ such that

(b) $\int\limits_{R^n} dx \sup\limits_{|t-t_0|<\partial} |\frac{\partial g}{\partial t}(t,x)| \quad < \infty$

Then

$\frac{d}{dt} \int dx\ g(t,x) : \int dx\ \frac{\partial}{\partial t} g(t,x)$ at $t = t_0$

3. (a) For any function $w : R^2 \to \mathbb{C}$ of the form $w(x,y) = f(x+y) + g(x-y), f, g : R \to \mathbb{C}$, show that $w(x,y) + w(x+\xi-\eta,\ y+\xi+\eta) = w(x+\xi,\ y+\xi) + w(x-\eta,\ y+\eta)$ for all real x, y, ξ, η. [3]

   (b) Let $u : R^2 \to \mathbb{C}$ be any C^2 function satisfying the wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0.$$

   Show that there exist functions $f_1, f_2 : R \to \mathbb{C}$ such that $u(x,y) = f_1(x+y) + f_2(x-y)$ [3]

4. Let $h : R \to R$ be a $C\ '$ function and $u : R^2 \to R$ satisfy $u_x + u_y = u, \quad u(x,0) = h(x)$. solve for u. [6]

5. Let p, q be real constants. Let Q be any solution g

$$y'' + py' + qy = 0$$

   (E)

   (a) If $p, q > 0$ then $\lim\limits_{x\to\infty} \phi(x) = 0$ [3]

   (b) If every solution ϕ of (E) satisfies $\lim\limits_{x\to\infty} \phi(x) = 0$ show that $p, q > 0$ [4]

6. Let a, b, c be reals with $a > 0, c > 0$. Show that every solution of the equation

$$\frac{dy}{dt} + ay = be^{-et}$$

approaches 0 as t approaches ∞ [6]

7. Let $f : R \to R$ be given by $f(x) = \log(1 + e^x)$

   (a) Show that $|f(x) - f(y)| \leq |x - y|$
   (b) Show that f has no fixed point
   (c) Show that $\sup_{x \neq y} \left| \frac{f(x)-f(y)}{x-y} \right| = 1$ [3]